# ON THE SOLUTION AND CHARACTERISTIC EXPONENTS OF SOLUTIONS OF SOME SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

**(O RESHENII I KHARAKTERICHESKIKH POKAZATELIAKH RESHENII NEKOTORYKH SISTEM LINEINYKH DIFFERENTSIAL'NYKH URAVNENII S PERIODICHESKIMI KOEFFITSIENTAMY)**

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A method is presented for the solution of some systems of linear differential equations with periodic coefficients by the use of the Laplace transform. The obtained results are used to find the characteristic exponents of the solutions of systems closely related to the given ones.

A criterion is given for the stability of the solutions of a second-order equation with periodic coefficients for the case of resonance.*

Of the classical methods in the theory of linear differential equations with periodic coefficients that are most closely related to the present work there must be mentioned the method of Ince [1] (p. 32) for the solution of Mathieu's equation by means of continued fractions (see Section 9).

1. In the sequel, capital letters will be used to denote matrices; $E$ will stand for the unit matrix. The matrix $A(p) = \|a_{sj}(p)\|_1^m$ will be holomorphic (meromorphic and so on) in the region $\Sigma$ if the $a_{sj}(p)$ $(s, j = 1, \ldots, m)$ are holomorphic (meromorphic and so on) in the region $\Sigma$. We denote the norm of a matrix by

$$|A| = m \max|a_{js}| \quad (j, s = 1, \ldots, m), \quad |A(p)|_{p\in\Sigma} = \sup|A(p)| \ (p\in)\Sigma. \tag{1.1}$$

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By the residue of the matrix $A(p)$ at the point $p = p_0$ shall mean the matrix of the residues of the elements and we will denote it by $\operatorname{res} A(p_0)$.

Let us apply the Laplace transformation [2] to the matrix $Y(t)(t \geqslant 0)$. This means that we apply the Laplace transformation to each element of the matrix $Y(t)(t \geqslant 0)$. The correspondence between the original $Y(t)$ and the image $F(p)$ extended by means of analytic continuation over the entire region of existence, will be denoted by

$$Y(t) \leftarrow\!\!\div F(p) \tag{1.2}$$

Many properties of the Laplace transform can be carried over to matrices. We note that if $R$ is a constant $m \times m$ matrix with characteristic numbers $p_1, \ldots, p_m$, then

$$e^{Rt} \leftarrow\!\!\div (pE - R)^{-1} \tag{1.3}$$

The matrix $(pE - R)^{-1}$ can have poles at the points $p_1, \ldots, p_m$ only.

**2.** In Sections 2 and 3 we consider the general form and some properties of the Laplace transform of the solutions of a homogeneous and of a nonhomogeneous system of linear differential equations with periodic coefficients.

Let us consider the homogeneous system of differential equations with periodic coefficients of the form

$$\frac{dY}{dt} + A(t)\,Y = 0 \qquad (A(t) = \| a_{sj} \|_1^m,\ A(t + 2\pi) = A(t)) \tag{2.1}$$

where the matrix $A(t)$ is a complex function of bounded variation on $[0, 2\pi]$. The fundamental matrix $N(t)$ of the solutions of the system (2.1) is representable in the form [3] (p.90)

$$N(t) = P(t)\,e^{Rt} \qquad \left(P(t) = \sum_{k=-\infty}^{\infty} P_k e^{ikt}\right) \tag{2.2}$$

Here $R$, $P_k$ are constant $m \times m$ matrices. The characteristic numbers $p_1, \ldots, p_m$ of the matrix $R$ are the characteristic exponents of the solutions of the system (2.1).

If $N(t) \leftarrow\!\!\div F(p)$, then (1.3) yields

$$F(p) = \sum_{k=-\infty}^{\infty} P_k\,(E_p - R - ikE)^{-1} \tag{2.3}$$

Since the order of $|P_k|$ is not higher than $k^{-2}$ [4], the series (2.3) converges. Hence we have the following result.

*Lemma 2.1.* The image (Laplace transform) of the fundamental matrix of the solution of the system (2.1) of linear differential equations with periodic coefficients is analytically extendible over the entire plane of the complex variable $p$, and is a meromorphic matrix $p$ whose poles can be only at the points

$$p_{sk} = p_s - ki \qquad (s = 1, \ldots, m,\ k = 0, \pm 1, \pm 2, \ldots) \tag{2.4}$$

where the $p_s (s = 1, \ldots, m)$ are the characteristic exponents of the solutions of system (2.1). Denoting by $\Sigma_\epsilon$ the entire plane of the complex variable $p$ except for the regions $|p - p_{sk}| < \epsilon$ $(\epsilon < 0)$, we obtain

$$|F(p)| \to 0 \qquad \text{as } p \to \infty \text{ with } p \in \Sigma_\epsilon$$

***Note 2.1.*** Let the poles of the matrix $F(p)$ lie in the left half-plane, or if they lie on the axis Re $p = 0$, then let them be of the first order. Under these conditions the solutions of the system (2.1) are stable. In the opposite case, they are unstable. This follows from the relations (2.2) and (2.3).

**3.** Let us find the particular solution of the nonhomogeneous linear differential equation

$$\frac{dY}{dt} + A(t)\,Y = \Phi(t) \tag{3.1}$$

The left-hand side of the system (3.1) coincides with that of (2.1). Let us suppose that

$$\Phi(t) \div Q(p) \tag{3.2}$$

It is assumed that the image $Q(p)$ exists and is holomorphic when Re $p >= b_0 =$ const. In our particular case we assume that

$$\Phi(t) = \sum_{j=1}^{\lambda} C_j t^{\nu_j} e^{\omega_j t} \div Q(p) = \sum_{j=1}^{\lambda} C_j \nu_j! \,(p - \omega_j)^{-\nu_j - 1} \tag{3.3}$$

where $C_j$ are constant matrices, $\nu_j$ is a negative integer, $\omega_j$ is a complex number. We have [ 3 ] (p.86)

$$Y(t) = \int_0^t N(t)\, N^{-1}(\tau)\, \Phi(\tau)\, d\tau \tag{3.4}$$

The matrix $N^{-1}(\tau)$ can be represented in the form

$$N^{-1}(\tau) = e^{-R\tau} \sum_{k=-\infty}^{\infty} M_k e^{ik\tau} \tag{3.5}$$

Substituting (3.5) and (2.2) into (3.4), we obtain

$$Y(t) = \sum_{r=-\infty}^{\infty} e^{-irt} \int_0^t \sum_{k=-\infty}^{\infty} P_{-r+k} \exp\{(t-\tau)(R+ikE)\} M_{-k}\Phi(\tau)\,d\tau \quad (3.6)$$

By the use of the theorem on multiplication and translation [2] (p. 474), we obtain on the basis of 3.2 the following result.

$$Y(t) \div \sum_{r=-\infty}^{\infty} B_r(p)\, Q(p+ri) \quad (3.7)$$

where

$$B_r(p) = \sum_{k=-\infty}^{\infty} P_k\,(Ep - R - ikE)^{-1} M_{-k-r} \quad (3.8)$$

*Lemma 3.1.* The matrices $B_r(p)$ can be extended over the entire plane of the complex variable $p$ and are meromorphic matrices of $p$. Their poles can occur only at the points $p_{sk}$ (2.4). In the region

$$\Sigma_\varepsilon (|p - p_{sk}| \geqslant \varepsilon, \qquad (\varepsilon > 0,\ s = 1, \ldots, m,\ k = 0, \pm 1, \pm 2, \ldots)$$

we have

$$|B_r(p)| = O(r^{-1}) \quad (r \to \pm\infty), \qquad |B_r(p)| \underset{p\to\infty}{\to} 0 \qquad \text{when } p \in \Sigma_\varepsilon$$

For the proof of this lemma it is sufficient to note that $|M_k| = O(k^{-2})$.

***Note 3.1.*** Let us consider the equations in the $B_r(p)$ with $Q(p)$ from (3.2)

$$\sum_{r=-\infty}^{\infty} B_r(p)\, Q(p+ri) = 0 \quad (3.9)$$

where the matrices $B_r(p)$ are bounded and holomorphic when $|\mathrm{Re}\ p| > b_1 =$ const. Let us substitute $Q(p)$ in this equation in the particular case when

$$Q(p) = E\,(p - p_0)^{-2} \div Ete^{p_0 t}, \qquad |\mathrm{Re}\ p| \geqslant b_1 \quad (3.10)$$

Since the series (3.9) converges when $p \approx p_0 - ri$; $B_r(p_0 - ri) = 0$. Because of the arbitrariness of $p_0$ and of the holomorphness of $B_r(p)$ we have $B_r(p) \equiv 0$. Hence, the image of $Y(t)$ in the form (3.7) (where the $B_r(p)$ are bounded and holomorphic when $|\mathrm{Re}\ p| > b_1 =$ const) is unique.

*Lemma 3.2.* If $\Phi(t)$ has the form (3.3), then the image (3.7) of $Y(t)$ can be extended over the entire plane of the complex variable and it is a meromorphic matrix whose poles can occur only at the points $p_{sk}$ (2.4) and at the points

$$\omega_{jk} = \omega_j - ki \qquad (j = 1, \ldots, \lambda;\ k = 0, \pm 1, \pm 2, \ldots) \tag{3.11}$$

In the region

$$\Sigma_\varepsilon^\circ \qquad (|p - p_{sk}| \geqslant \varepsilon,\ |p - \omega_{jk}| \geqslant \varepsilon;\ \varepsilon > 0)$$
$$(s = 1, \ldots, m;\ j = 1, \ldots, \lambda;\ k = 0, \pm 1, \pm 2, \ldots)$$

the series (3.7) converges absolutely and uniformly and

$$|F(p)| \to 0 \quad \text{when } p \to \infty \quad \text{and } p \in \Sigma_\varepsilon^\circ$$

The proof follows from Lemmas 2.1 and 3.2.

**4.** Let us consider the system of linear differential equations of the particular form

$$\sum_{q=-l}^{l} e^{-iqt} L_q(d)\, Y(t) = \Phi(t) \qquad \left(d = \frac{d}{dt}\right) \tag{4.1}$$

where the $L_q(d)$ are linear differential operators

$$L_q(d) = \sum_{j=0}^{n} A_{qj} d^j \qquad (q = 0, \pm 1, \pm 2, \ldots) \tag{4.2}$$

The $A_{qj}$ are constant $m \times m$ matrices. Let us suppose that

$$A_{0n} \equiv E, \qquad \sum_{\substack{q=-l \\ q \neq 0}}^{l} |A_{qn}| < 1 \tag{4.3}$$

For the sake of simplicity we assume that $l$ is finite. If $l = \infty$, it is sufficient to require that the following conditions be satisfied:

$$\sum_{q=-\infty}^{\infty} |A_{qj} q^j| < \infty \qquad (j = 0, 1, \ldots, n) \tag{4.4}$$

We are looking for a solution of the system (1.1) with the initial conditions

$$Y(0) = Y_0^{(0)}, \ldots, Y^{(n-1)}(0) = Y_0^{(n-1)} \tag{4.5}$$

Applying the Laplace transformation to the system (4.1) with $t > 0$, we obtain a system of linear difference equations for the image $F(p)$ of the solution of the system (4.1) with the initial conditions (4.5)

$$\sum_{q=-l}^{l} L_q(p + qi)\, F(p + qi) = Q(p) + \sum_{q=-l}^{l} \Psi_q(p + qi) \tag{4.6}$$

where

$$\Psi_q(p) = \sum_{k=0}^{n-1} \sum_{j=k+1}^{n} A_{qj} Y_0^{(k)} p^{j-k-1} \tag{4.7}$$

Let us introduce the notation

$$\Omega(p) = L_0^{-1}(p)\Big(Q(p) + \sum_{q=-l}^{l} \Psi_q(p+qi)\Big) \tag{4.8}$$

$$K_q(p) = -L_0^{-1}(p) L_q(p+qi), \qquad \tau(p) = \sum_{\substack{q=-l \\ q\neq 0}}^{l} |K_q(p)| \tag{4.9}$$

From (4.7), (4.8), (4.2) and (4.9) it follows that

$$\Omega(p) \to 0 \text{ when } \operatorname{Re} p \to +\infty, \qquad K_q(p) \to -A_{qn} \text{ when } p \to \infty \; (q = \pm 1, \ldots, \pm l). \tag{4.10}$$

The system (4.6) of difference equations takes on the form

$$F(p) = \Omega(p) + \sum_{\substack{q=-l \\ q\neq 0}}^{l} K_q(p) F(p+qi) \tag{4.11}$$

Suppose $F(p)$ is a solution of the system (4.11) when $\Omega(p) \equiv 0$. And suppose that this solution is bounded and holomorphic when $\operatorname{Re} p \geqslant b_0 =$ const. From (4.10) and (4.3) it follows that there exists such a number $b_1 \geqslant b_0$ that $\sup \tau(p) < 1$ when $\operatorname{Re} p \geqslant b_1$.

Let $|F(p)|_{\operatorname{Re} p > b_1} = a (0 < a < \infty)$. Then it follows from (4.11) that $|F(p)|_{\operatorname{Re} p \geqslant b_1} < a$, i.e. $F(p) \equiv 0$.

The existence of a bounded solution $F(p)$ of the system of equations (4.1) when $\Omega(p) \not\equiv 0$ follows from Lemmas 2.1 and 3.2.

We have thus proved the following lemma.

*Lemma 4.1.* If $F(p) \doteq Y(t) (t > 0)$, where $Y(t)$ is the solution of the system (4.1) with the initial conditions (4.5), then $F(p)$ satisfies the system of linear difference equations (4.6), (4.11), which has a unique holomorphic solution bounded when $\operatorname{Re} p > b_0$, where $b_0$ is large enough.

***Note 4.1.*** Many systems of linear differential equations with isolated regular and nonregular singular points can be reduced [5, 6] to the form (4.1) by means of a change of the independent variable.

We give a few particular cases of finding the original from the image, and of the construction of the representation.

**5.** In the particular case, when $\Phi(t)$ has the form (3.3), it can be shown that

$$Y(-t) \leftarrow\!\!\div -F(-p) \qquad (t>0) \quad \text{when } Y(t) \leftarrow\!\!\div F(p) \ (t>0)$$

From this it follows that

$$Y(t) = \frac{1}{2\pi i}\,\text{v. p.}\int_{b_1-i\infty}^{b_1+i\infty} e^{pt}F(p)\,dp + \frac{1}{2\pi i}\,\text{v. p.}\int_{-b_1+i\infty}^{-b_1-i\infty} e^{pt}F(p)\,dp \qquad (5.1)$$

where $b_1$ is sufficiently large. From Lemmas 2.1 and 3.2, we obtain

$$Y(t) = \text{v.p.}\sum_j \operatorname{res} F(p_j)\, e^{p_j t} \qquad (5.2)$$

where the summation is carried out over the poles of the matrix $F(p)$. For finite $l$ in (4.1), the series (5.2) converges in some strip along the real axis $t$. If in (4.2) the matrices $A_{qn} \equiv 0 (q = \pm 1, \ldots, \pm 1)$, then the series (5.2) converges absolutely and uniformly in an arbitrary finite region of the complex variable $t$.

Let us consider the characteristic equation of the system (4.1) averaged with respect to time,

$$\operatorname{Det} L_0(p) = 0 \qquad (5.3)$$

Here, and in the sequel, we denote by $\rho_1, \ldots, \rho_{m\times n}$ the roots of the equations

$$\rho_{jk} = \rho_j - ki \qquad (j = 1, \ldots, m\times n,\ k = 0, \pm 1, \pm 2, \ldots) \qquad (5.4)$$

(a) When $L_q(p) \equiv 0$ $(q = \pm 1, \ldots, \pm 1)$, we have a system of linear differential equations with constant coefficients. Such equations are usually solved by the use of the Laplace transform.

(b) Let $L_q(p) \equiv 0$ $(q = -1, \ldots, -1)$. In this case one can find the needed solution of the system of linear difference equations by the method of successive approximations from the recursion formulas

$$F_0(p) = \Omega(p), \quad F_j(p) = \Omega(p) + \sum_{q=1}^{l} K_q(p)\, F_{j-1}(p+qi) \quad (j = 1, 2 \ldots) \qquad (5.5)$$

so that $F_j(p) \to F(p)$ as $j \to \infty$, where $F(p)$ is a meromorphic matrix and its poles can occur only at the points $\rho_{jk}$ (5.4) and $\omega_{jk}$ (3.11) for negative values of $k$.

In the given case this method is similar to the method of Frobenius for the solution of a system of linear equations of the Fuchsian type with a regular singular point [3].

**6.** The sections 6 and 7 are the most essential ones in this work. In them we study the properties of some auxiliary matrix functions. These functions are then used for the construction of the analytic continuation of the solution of the system of linear difference equations (4.11).

Let $r_0, r_1, \ldots, r_\beta, k_0, k_1, \ldots, k_\alpha$ be integers. We introduce the matrix functions

$$S(p) = S^{r_0, r_1, \ldots, r_\beta}_{k_0 k_1, \ldots, k_\alpha}(p) = \sum_{\sigma=1}^{\infty} \sum_{q_1 \ldots q_\sigma}' K_{q1} K_{q2} \cdots K_{q_\sigma}$$

$$K_{q_1} = K_{q_1}(p + k_0 i), \quad K_{q_2} = K_{q_2}(p + (k_0 + q_1) i), \ldots,$$

$$K_{q\sigma} = K_{q\sigma}(p + (k_0 + q_1 + \ldots + q_{\sigma-1}) i)$$

$$(q_1 = \pm 1, \ldots, \pm l; \ldots; q_\sigma = \pm 1, \ldots, \pm l) \tag{6.1}$$

Here the superscript prime (′) at the summation symbol indicates that one takes only those terms whose indices $q_1, \ldots, q_0$ satisfy the additional conditions

(a) $k_0 + q_1 + q_2 + \cdots + q_\sigma = r_0$

(b) $\{k_1, \ldots, k_\alpha\} \cap \{k_0 + q_1, k_0 + q_1 + q_2, \ldots, k_0 + q_1 + q_2 + \cdots + q_{\sigma-1}\} = \Lambda$

(c) $\{r_1, \ldots, r_\beta\} \subset \{k_0 + q_1, k_0 + q_1 + q_2, \ldots, k_0 + q_1 + q_2 + \cdots + q_{\sigma-1}\}$

Here, and in the sequel the following notations are used: $\{\ldots\}$ indicates a set; $\cap$ denotes the intersection of sets; $\subset$ is the inclusion sign for sets; $\Lambda$ is the empty set, [null set]; $K_q(p)$ is the matrix in (4.9).

If for an arbitrary $\sigma > 0$ there does not exist a set of indices $q_j (q_j = \pm 1, \ldots, \pm 1, j = 1, \ldots, \sigma)$ such that the conditions (*a*), (*b*), (*c*) are simultaneously satisfied, then we set

$$S(p) = 0 \tag{6.2}$$

Let us consider a finite closed region $\Sigma$ of the complex plane $p$. We take notice of those values of $k$ for which the points $\rho_{ik}$ (5.4) lie in the region $\Sigma$, and we denote these values of $k$ by $k_1, \ldots, k_\alpha$.

*Lemma 6.1.* If $k_0 \in \{k_1, \ldots, k_\alpha\}$ then for sufficiently small $|A_{qj}|$ $(q = \pm 1, \ldots, \pm 1, j = 0, 1, \ldots, n)$ the series (6.1) converges absolutely in the region $\Sigma$ to a regular matrix with an arbitrarily small norm.

*Proof*. Let us denote by $\chi$ the sup $r(p + ki)$ with $k \neq k_1, \ldots, k_\alpha$ and $p \in \Sigma$. From (4.9) it follows that if the $|A_{qj}|$ $(q = \pm 1, \ldots, \pm 1,$ $j = 0, 1, \ldots, n)$ are sufficiently small, then the $r(p)$ (and with it $\chi$) will be as small as we please when $|p - \rho_j| > \epsilon$ $(\epsilon > 0,\ j = 1, \ldots, m \times n)$.

Since

$$|\Sigma' K_{q_1} K_{q_2}, \ldots, K_{q_\sigma}| \leqslant \sum_{q_1}' |K_{q_1}| \sum_{q_2}' |K_{q_2}|, \ldots, \sum_{q_\sigma}' |K_{q_\sigma}| \leqslant \chi^\sigma \qquad (6.3)$$
$$(q_1 = \pm 1, \ldots, \pm l; \ldots; q_\sigma = \pm 1, \ldots, \pm l)$$

where

$$K_{q_1} = K_{q_1}(p + k_0 i), \qquad K_{q_2} = K_{q_2}(p + (k_0 + q_1)\,i), \ldots$$
$$K_{q_\sigma} = K_{q_\sigma}(p + (k_0 + q_1 + \ldots + q_{\sigma-1})\,i)$$

the norms of the terms of the series (6.1) will be dominated by the terms of a geometric series. The matrices $K_q(p + ki)$ are regular in the region $\Sigma$ with $k \neq k_1, \ldots, k_\alpha$. Therefore, the series (6.1) converges when $\chi < 1$ to a regular matrix whose norm is arbitrarily small for small enough values of $\chi$.

***Note 6.1***. Since if follows from (4.9), (4.3) and (4.4) that

$$\lim_{p \to \infty} \tau(p) = \sum_{\substack{q=-l \\ q \neq 0}}^{l} |A_{qn}| < 1 \qquad (6.4)$$

we have (for Re $p > b$, where $b$ is large enough) that the series (6.1) converges absolutely to a regular bounded matrix. Furthermore, we assume that the matrices $S$, determined by (6.1), are continued analytically from the region Re $p > b$ over the entire region of their existence.

***Note 6.2***. Let $\Sigma$ be bounded region of the complex plane $p$. From (6.3) it follows that there exist integers $k_1 < k_2$ such that the series (6.1) for the matrix

$$S^{r_0, r_1, \ldots, r_\beta}_{k_0, k_1, k_1+1, \ldots, k_2}(p) \quad \text{when } k_0 \in [k_1, k_2]$$

converges if $p \in \Sigma$.

***Note 6.3***. If in the lemma 5, $k_0 \in (k_1, \ldots, k_\alpha)$, then the series (6.1) will have terms containing the factors $K_q(p + k_0 i)$ (4.9) which can have poles in the region $\Sigma$. If one wants to eliminate these singularities it is sufficient to multiply the series (6.1) on the left by $L_0(p + k_0 i)$.

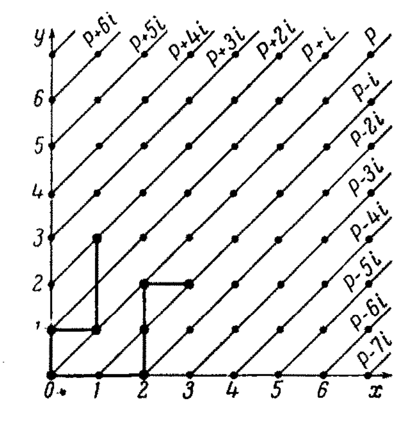
From this it follows that the matrix

$$L_0(p + k_0 i)\, S^{r_0, r_1, \dots, r_\beta}_{k_0, k_1, \dots, k_\alpha}(p) \quad \text{when } p \in \Sigma$$

is regular and arbitrarily small in norm for small enough $|A_{qj}|$ $(q = \pm 1, \dots, \pm 1,\ j = 0, 1, \dots, n)$, and the series (6.1) converges when $p \in \Sigma$ but $p \neq \rho_j - k_0 i$ $(j = 1, \dots, m \times n)$.

7. At the beginning of this section, we give a geometric interpretation of the matrix functions $S(p)$ defined by (6.1). This interpretation is convenient for the study of the properties of the matrix functions.

Let us consider the first quadrant of rectangular coordinates (Fig. 1). To every diagonal (straight line $y = x + k$, where $k$ is an integer) there corresponds the argument $p + ki$ indicated at the upper right corner. In Fig. 1 we construct the broken lines which consist of the vertical and horizontal segments whose lengths are integers not greater than the number 1.




The beginning of a broken line is a point on either the $x$-axis or the $y$-axis with integer coordinates. We place at this point the initial point of the first segment in such a way that the segment is directed either vertically upward or horizontally to the right, and so on.

The broken lines may consist of one, two or more vertical and horizontal segments.

We consider two broken lines to be distinct if they do not coincide along all of their segments. If the beginning of the first segment lies on the diagonal $p + k_0 i$, then we shall say that the broken line begins on the diagonal $p + k_0 i$; if the end of the last segment of a given broken line lies on the diagonal $p + k_0 i$, then this broken line is said to terminate, or end on the diagonal $p + r_0 i$.

Suppose that a given broken line consists of $\sigma > 1$ segments. If at least one of the ends of the segments $1, \dots, \sigma - 1$ lies on the diagonal $p + si$ then we shall say that the given broken line intersects the diagonal $p + si$; in the opposite case we shall say that this broken line intersect this diagonal.

We associate with each of the described broken lines an expression. This expression shall be the product of the matrices $K_q(p + si)$. To every

*Property 7.1.* Let $s$ be an integer. Then

$$S(p) = S^{r_0, r_1, \ldots, r_\beta}_{k_0, k_1, \ldots, k_\alpha}(p) = S^{r_0+s, r_1+s, \ldots, r_\beta+s}_{k_0+s, k_1+s, \ldots, k_\alpha+s}(p - is) \tag{7.1}$$

*Property 7.2.* If some of the number $k_1, \ldots, k_\alpha$ and $r_1, \ldots, r_\beta$ are equal, then $S(p) \equiv 0$ because it is impossible to satisfy simultaneously conditions $c$ and $d$ of Section 7 (or conditions $b$ and $c$ of Section 6).

*Property 7.3.* Let $k_0 \neq r_0$. Let us lay off on the real axis the numbers $k_0, k_1, \ldots, k_\alpha r_0$. If between $k_0$ and $r_0$ there will lie at the integer points not less than $l$ numbers of the numbers $k_1, \ldots, k_\alpha (a \geqslant 1)$, then condition (6.2) is satisfied.

*Property 7.4.* Let $s$ be an integer and $s \overline{\in} (k_1, \ldots, k_\alpha, r_1, \ldots, r_\beta)$, then

$$S^{r_0, r_1, \ldots, r_\beta}_{k_0, k_1, \ldots, k_\alpha}(p) = S^{r_0, r_1, \ldots, r_\beta}_{k_0, k_1, \ldots, k_\alpha, s}(p) + S^{r_0, r_1, \ldots, r_\beta, s}_{k_0, k_1, \ldots, k_\alpha}(p) \tag{7.2}$$

For the proof we take from the series standing on the left-hand side of Equation (7.2) an arbitrary term, that is, a matrix product

$$K_{q_1}(p + k_0 i) K_{q_2}(p + (k_0 + q_1) i) \ldots K_{q_\sigma}(p + (k_0 + q_1 + q_2 + \ldots + q_{\sigma-1}) i)$$

For a given $k_0$ this term will be completely determined by the indices $q_1, \ldots, q_\sigma (q_j = \pm 1, \ldots, \pm 1, j = 1, \ldots, \sigma)$. We separate the set of different sequences of the indices $q_1, \ldots, q_\sigma$ satisfying conditions (a), (b), (c) and (d) into two nonoverlapping groups. (The terms of the series (6.1) are hereby also separated into two nonoverlapping groups.) The sequence of indices $q_1, \ldots, q_\sigma$ is put into the first group if

$$s \overline{\in} \{k_0 + q_1, k_0 + q_1 + q_2, \ldots, k_0 + q_1 + q_2 + \ldots + q_{\sigma-1}\}$$

and into the second group, if

$$s \in \{k_0 + q_1, k_0 + q_1 + q_2, \ldots, k_0 + q_1 + q_2 + \ldots + q_{\sigma-1}\}$$

Conversely, from the conditions (a), (b), (c), (6.1) it follows that every term of the series on the right-hand side of (7.2) is also contained in the series of the left-hand side of (7.2).

In the geometric notation this property implies that all admissible broken lines are divided into two classes: those which intersect the diagonal $p + si$, and those which do not intersect it.

*Property 7.5.* From the definitions given in Sections 6 and 7 it follows that

$j$th segment there corresponds a particular factor $K_q(p + s_j i)(j = 1, \ldots, \sigma)$. Here $|q|$ is equal to the length of the segment ($q = \pm 1, \ldots, \pm 1$). If the segment is directed upward, $q > 0$; if it is directed to the right, $q < 0$. The number $p + s_j i$ is the argument of the diagonal on which the $j$th segment of the given broken line begins. The order of succession of the segment determines the order of the factors which are multiplied from the right.

We call attention to the fact that a given factor corresponds to different segments in Fig. 1, which differ only in a translation parallel to the bisector of the principal angle.

*Example 1.* To the broken lines of Fig. 1 there correspond the following expressions

$$K_1(p)\,K_{-1}(p+i)\,K_2(p), \qquad K_{-2}(p)\,K_1(p-2i)\,K_1(p-i)\,K_{-1}(p)$$

*Definition 7.1.* Let $k_0, k_1, \ldots, k_\alpha, r_0, r_1, \ldots, r_\beta$ be integers. We select the broken lines satisfying the conditions:

(a) The beginning of the broken lines is on the diagonal $p + k_0 i$.

(b) The end of the broken line is on the diagonal $p + r_0 i$.

(c) The broken lines do not intersect the diagonals $p + k_1 i$, $p + k_2 i, \ldots, p + k_\alpha i$.

(d) Each broken line intersects the diagonals $p + r_1 i$, $p + r_\alpha i, \ldots, p + r_\beta i$.

The sum of the expressions that correspond to all possible distinct broken lines satisfying the conditions (a), (b), (c), and (d) we denote by

$$S_{k_0, k_1, \ldots, k_\alpha}^{r_0, r_1, \ldots, r_\beta}(p) = S(p)$$

This definition agrees with the earlier given one (6.1).

From the condition (d) it follows that $\sigma$ (the number of segments making up an admissible broken line) cannot be smaller than $\beta + 1$ if $r_1, \ldots, r_\beta$ are distinct.

On the basis of our geometric interpretation, let us consider the properties of the introduced matrix functions when $\operatorname{Re} p > b$, where $b$ is large enough. Since the series of the type (6.1) are absolutely convergent when $\operatorname{Re} p > b$ (Note 6.1), one may change the order of summation and rearrange the terms of the series. Because of the uniqueness of the analytic continuation of a function, the properties listed below will apply also to the analytically extended matrices.

$$S^{r}_{k_0, k_1, \dots, k_\alpha}(p) = \Sigma' K_q(p + k_0 i) S^{r}_{q+k_0, k_1, \dots, k_\alpha}(p) + \Sigma \delta^{r}_{q+k_0} K_q(p + k_0 i) \quad (7.3)$$
$$(q = \pm 1, \dots, \pm l; \quad q + k_0 \neq k_1, \dots, k_\alpha)$$

The superscript prime (′) indicates that we have omitted the terms which do not satisfy the additional conditions

$$\delta_s^{\ r} = 1 \text{ if } r = s, \qquad \delta_s^{\ r} = 0 \text{ if } r \neq s$$

In the last sum there stands out one term which corresponds to the single segment of length $|r - k_0|$.

*Property 7.6.* In analogy with property 7.5 we may write

$$S^{r}_{k_0, k_1, \dots, k_\alpha}(p) = \Sigma' S^{r+q}_{k_0, k_1, \dots, k_\alpha}(p) K_{-q}(p + (r + q) i) + \Sigma \delta^{r}_{q+k_0} K_q(p + k_0 i)$$
$$(q = \pm 1, \dots, \pm l; \ q + r \neq k_1, \dots, k_\alpha) \quad (7.4)$$

One may also write down relations analogous to (7.5) and (7.6) in case $\beta \neq 0$.

*Property 7.7.* If $r_1 \neq k_1, \dots, k_\alpha$, then

$$S^{r_0, r_1}_{k_0, k_1, \dots, k_\alpha}(p) = S^{r_1}_{k_0, k_1, \dots, k_\alpha, r_1}(p) S^{r_0}_{r_1, k_1, \dots, k_\alpha}(p) \quad (7.5)$$

*Proof.* We denote by $q_1', \dots, q_n'$, and $q_1', \dots, q_\sigma'$, the indices which determine the terms (see property 7.4) in the first and second series on the right-hand side of (7.5). From the definition (6.1) it follows that the following conditions are satisfied:

$$k_0 + q_1' + \dots + q_{\sigma'}' = r_1, \quad \{k_1, \dots, k_\alpha, r_1\} \cap$$
$$\cap \{k_0 + q'_1, \ k_0 + q_1' + q_2', \dots, k_0 + q_1' + q_2' + \dots + q'_{\sigma'-1}\} = \Lambda \quad (7.6)$$
$$r_1 + q_1'' + \dots + q_{\sigma''}'' = r_0, \quad \{k_1, \dots, k_\alpha\} \cap$$
$$\cap \{r_1 + q_1'', \ r_1 + q_1'' + q_2'', \dots, r_1 + q_1'' + q_2'' + \dots + q''_{\sigma''-1}\} = \Lambda$$

From this it follows that the sequence of indices $q_1', \dots, q_{\sigma'}'$, $q_1'', \dots, q_{\sigma''}''$ satisfies the conditions

(*a*) $k_0 + q_1' + \dots + q_{\sigma'}' + q_1'' + \dots + q_{\sigma''}'' = r_0$

(*b*) $\{k_1, \dots, k_\alpha\} \cap \{k_0 + q_1', \dots, k_0 + q_1' + \dots + q'_{\sigma'-1}, r_1, r_1 + q_1'', \dots$
$\dots, r_1 + q_1'' + \dots + q''_{\sigma''-1}\} = \Lambda$

(*c*) $r_1 \in \{k_0 + q_1', \dots, k_0 + q_1' + \dots + q'_{\sigma'-1}, r_1, r_1 + q_1'', \dots,$
$r_1 + q_1'' + \dots + q''_{\sigma''-1}\}$

that is, the product of two arbitrary terms in the series on the right of (5.7) enters also into the left side of (7.5). Conversely, from (6.1) it follows that every sequence of indices $q_1, \ldots, q_\sigma$, which determines a term on the left side of (7.5), can be broken up in a unique manner into groups

$$q_j = q_j' \quad (j = 1, \ldots, \sigma'), \qquad q_{j+\sigma'} = q_j'' \quad (j = 1, \ldots, \sigma'') \qquad (\sigma = \sigma' + \sigma'')$$

so that $q_j'$ and $q_j''$ satisfy the conditions (7.6). From this it follows that an arbitrary term of the left side of (7.5) can be uniquely represented as the product of two terms which enter respectively into the first and second factors of the right side of (7.5).

The property 7.7 is thus established. In the geometric interpretation this property follows from the fact that each admissible broken line must intersect the diagonal $p + r_1 i$ at least once. The part of the broken line which precedes the first intersection with the diagonal $p + r_1 i$ corresponds to the first factor in (7.5). The essential role is played by the circumstance that the translation of the broken line parallel to the bisector of the principal angle does not affect the form of the expression corresponding to the given broken line. Analogously, by separating the broken line into parts preceding and following the intersection of this line with the diagonal $p + r_1 i$, we obtain

$$S_{k_0, k_1, \ldots, k_\alpha}^{r_0, r_1}(p) = S_{k_0, k_1, \ldots, k_\alpha}^{r_1}(p) S_{r_1, k_1, \ldots, k_\alpha, r_1}^{r_0}(p) \tag{7.7}$$

if $r_1 \neq k_1, \ldots, k_\alpha$.

*Property 7.8.* Let $k_0 \neq k_1, \ldots, k_\alpha$. From the properties 7.6 and 7.7 we obtain

$$S_{k_0, k_1, \ldots, k_\alpha}^{r}(p) = S_{k_0, k_1, \ldots, k_\alpha, k_0}^{r}(p) + S_{k_0, k_1, \ldots, k_\alpha}^{s, k_0}(p) =$$
$$= (E + S_{k_0, k_1, \ldots, k_\alpha}^{k_0}(p)) S_{k_0, k_1, \ldots, k_\alpha, k_0}^{r}(p) \tag{7.8}$$

*Property 7.9.* Let $r \neq k_1, \ldots, k_\alpha$. In analogy with property 7.7 we have

$$S_{k_0, k_1, \ldots, k_\alpha}^{r}(p) = S_{k_0, k_1, \ldots, k_\alpha, r}^{r}(p) + S_{k_0, k_1, \ldots, k_\alpha}^{r, r}(p) =$$
$$= S_{k_0, k_1, \ldots, k_\alpha, r}^{r}(E + S_{r, k_1, \ldots, k_\alpha}^{r}(p)) \tag{7.9}$$

*Property 7.10.* Let $\alpha = \beta = 0$, $k_0 = r_0 = 0$. From the properties 7.6 and 7.7 we obtain

$$S_0^0(p) = S_{0,0}^0(p) + S_{0,0}^0(p) = S_{0,0}^0(E + S_0^0(p))$$

Hence,

$$E + S_0^0(p) = (E - S_{0,0}^0(p))^{-1} \tag{7.10}$$

If $k_0 \neq k_1, \ldots, k_\alpha$, then in analogy with Formula (7.10) we obtain

$$E + S^{k_0}_{k_0, k_1, \ldots, k_\alpha}(p) = (E - S^{k_0}_{k_0, k_1, \ldots, k_\alpha, k_0}(p))^{-1} \qquad (7.11)$$

*Lemma 7.1.* Let $\Sigma$ be a bounded region of the complex plane. The properties 7.1 to 7.10 make it possible to continue analytically the matrix function $S^r_{k_0, k_1, \ldots, k_\alpha}(p)$ in $\Sigma$, that is, to express the given matrix function by means of a finite number of operations in terms of other matrix functions for which the series (6.1) converges in the region $\Sigma$.

*Proof.* From Notes 6.2 and 6.3 it follows that it is sufficient to express $S^r_{k_0, k_1, \ldots, k_\alpha}(p)$ in terms of matrix functions with an additional subscript $\gamma (\gamma \neq k_1, \ldots, k_\alpha)$. Let us consider the possible cases.

(A). Let $\gamma = k_0 \neq k_1, \ldots, k_\alpha$; from the properties 7.8 and 7.10 we obtain

$$S^r_{k_0, k_1, \ldots, k_\alpha}(p) = (E - S^{k_0}_{k_0, k_1, \ldots, k_\alpha, k_0}(p))^{-1} S^r_{k_0, k_1, \ldots, k_\alpha, k_0}(p) \qquad (7.12)$$

(B). Let $\gamma = r \neq k_0, k_1, \ldots, k_\alpha$; from the properties 7.9 and 7.10 it follows that

$$S^r_{k_0, k_1, \ldots, k_\alpha}(p) = S^r_{k_0, k_1, \ldots, k_\alpha, r}(p)(E - S^r_{r, k_1, \ldots, k_\alpha, r}(p))^{-1} \qquad (7.13)$$

(C). Let $\gamma \neq r, k_0, k_1, \ldots, k_\alpha$; from the properties 7.9, 7.10, and 7.12 $(\gamma = k_0)$ it follows that

$$S^r_{k_0, k_1, \ldots, k_\alpha}(p) = S^r_{k_0, k_1, \ldots, k_\alpha, \gamma}(p) +$$

$$+ S^\gamma_{k_0, k_1, \ldots, k_\alpha, \gamma}(p)(E - S^\gamma_{\gamma, k_1, \ldots, k_\alpha, \gamma}(p))^{-1} S^r_{\gamma, k_1, \ldots, k_\alpha, \gamma}(p) \qquad (7.14)$$

so that

$$S^{r, \gamma}_{k_0, k_1, \ldots, k_\alpha}(p) = S^\gamma_{k_0, k_1, \ldots, k_\alpha, \gamma}(p)\ S^r_{\gamma, k_1, \ldots, k_\alpha}(p) \qquad (7.15)$$

In each case it was possible to express $S^r_{k_0, k_1, \ldots, k_\alpha}(p)$ in terms of matrix functions with an additional subscript $\gamma$.

**8.** *Lemma 8.1.* A solution of the difference equations (4.11) is given by the series

$$F(p) = \Omega(p) + \sum_{r=-\infty}^{\infty} S_0{}^r(p)\, \Omega(p + ri) \qquad (8.1)$$

*Proof.* Let us substitute the series (8.1) into Equation (4.11) and equate the coefficients of $\Omega(p + ri)$. We obtain

$$S_0^0(p) = \sum_{q=\pm 1, \ldots, \pm l} K_q(p) S_0^{-q}(p+qi) \tag{8.2}$$

$$S_0^r(p) = \sum_{q=\pm 1, \ldots, \pm l} K_q(p) S_0^{r-q}(p+qi) + \sum_{q=\pm 1, \ldots, +l} \delta_q^r K_q(p) \quad (r \neq 0) \tag{8.3}$$

From the property 7.1 we have

$$S_0^{-q}(p+qi) = S_q^0(p), \qquad S_0^{r-q}(p+qi) = S_q^r(p) \quad (r \neq 0) \tag{8.4}$$

**and Equations (8.2) and (8.3) are satisfied identically on the basis of property 7.5.**

The series, which determine $S_0^r(p)$, diverge, as a rule, in the neighborhood of the points $\rho_{jk}$ (5.4). Making use of Lemma 7.1 one can analytically continue the series (8.1) over the region of interest, and thus study the behavior of the function. From the properties 7.1 to 7.10 we obtain

$$S_0^r(p) = (E - S_{0,0}^0(p))^{-1} S_{0,0}^r(p) \tag{8.5}$$

$$S_0^r(p) = S_{0,r}^r(p)(E - S_{0,0}^0(p+ri)) \tag{8.6}$$

Making the appropriate substitutions in (8.1) we have

$$F(p) = (E - S_{0,0}^0(p))^{-1}\left[\Omega(p) + \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} S_{0,0}^r(p)\Omega(p+ri)\right] \tag{8.7}$$

$$F(p) = (E - S_{0,0}^0(p))^{-1}\Omega(p) + \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} S_{0,r}^0(p)(E - S_{0,0}^0(p+ri))^{-1}\Omega(p+ri) \tag{8.8}$$

*Definition 8.1.* Solutions of the system of linear difference equations (4.11) of the form (8.1), (8.7) and (8.8) we shall call solutions of the first, second or third form respectively.

In the particular case when $\Phi(t)$ has the form (3.3), the locations of the points $\omega_{jk}$ (3.11) are known and, hence, one can compute the particular solution of the system of equations (4.1) by the use of Formula (5.2) without knowing the solution of the homogeneous system of equations (4.1).

**9.** Let us consider the determination of the characteristic exponents of the solution of a system close to the stationary system of equations (4.1).

From (4.8) and Note 3.1 it follows that

$$B_0(p) = (E - S_0^0(p))L_0^{-1}(p), \quad B_r(p) = S_0^r(p)L_0^{-1}(p + ri), \qquad r \neq 0 \tag{9.1}$$

For the computation of the characteristic exponents $p_{sk}$ it is sufficient to find the poles of the matrix $B_r(p)$. Let us consider the matrix $B_0(p)$.

*Lemma 9.1.* The roots of the equation

$$\text{Det}\,(L_0(p) - L_0(p)\,S_{0,0}^0(p)) = 0 \tag{9.2}$$

are the characteristic exponents of the solutions of the system of linear differential equations (4.1).

The proof follows from Formula (9.1), Lemma 3.1 and from the fact that

$$(E + S_0^0(p))L_0^{-1}(p) = (L_0(p) - L_0(p)\,S_{0,0}^0(p))^{-1} \tag{9.3}$$

Let us consider the case when the $A_{qj}$ $(q = \pm 1, \ldots, \pm 1,\ j = 0, 1, \ldots, n-1)$ contain a small parameter $\mu$,

$$A_{qj} = \mu B_{qj} \qquad (q = \pm 1, \ldots, \pm l,\ j = 0, 1, \ldots, n) \tag{9.4}$$

where the $B_{qj}$ are constant $m \times n$ matrices, while the $A_{0j}$ $(j = 0, 1, \ldots, n-1)$ are matrices and continuous functions of $\mu$. The characteristic exponents $p_{sk}$ (2.4) depend on $\mu$ (we shall write $p_{sk}(\mu)$). One can select $m \times n$ characteristic exponents (we shall denote them by $p_s(\mu)$) such that $p_s(0) = \rho_s$ $(s = 1, \ldots, m \times n)$ [see (5.3)]. Let us list some known properties of the $p_s(\mu)$.

*Property 9.1.* The exponents $p_s(\mu)$ are continuous functions of $\mu$.

*Property 9.2.* According to Liouville's theorem

$$\sum_{s=1}^{m\times n} p_s(\mu) = \sum_{s=1}^{m\times n} \rho_s$$

*Property 9.3. If*

$$A_{qj} = \overline{A}_{-qj} \qquad (q = +1, \ldots, +l,\ j = 0, 1, \ldots, n), \qquad \text{Im}\,A_{0j} = 0$$

then the $p_s(\mu)$ are distributed symmetrically relative to the axis $\text{Im}\,p = 0$.

*Property 9.4.* In canonical systems of linear differential equations the $p_r(\mu)$ are distributed symmetrically relative to the axis Re $p = 0$.

Let $p_*$ be some characteristic exponent of the solutions of the system (4.1) under the hypothesis (9.4) with $\mu = 0$.

We consider the sequence

$$\operatorname{Det} L_0 (p_* + ki) \qquad (\mu = 0,\ k = 0, \pm 1, \pm 2, \ldots) \tag{9.5}$$

The difficulty of finding $p_s(\mu)$ close to $p_*$ depends on the number of vanishing terms of the sequence (9.5).

1. Suppose in the sequence (9.5) only one term is zero (vanishes) when $k = 0$. Then the series for $L_0(p)S^0_{0,0}(p)$ converges for small enough $\mu$ in the region $|p - p_*| < \epsilon$ $(\epsilon > 0)$ and for an arbitrarily small norm (Lemma 6.1; Note 6.3). Since the broken lines that correspond to the term in $S^0_{0,0}(p)$ deviate from the diagonal $p$ and return again to it, we have

$$L_0 (p)\, S^0_{0,0} (p) = \mu^2 (\ )\ldots \tag{9.6}$$

According to a theorem of Rouche, the number of roots of (9.2) that are near to $p_*$, when $\mu$ is small enough, is equal to the multiplicity of the root $p_*$ of Equation (5.3).

*Consequence*. Let $\rho_1, \ldots, \rho_{m\times n}$ be the roots of Equation (5.3) and let

$$\rho_1 = \rho_2 = \cdots = \rho_h \neq \rho_j - ki \qquad (j = h+1, \ldots, m \times n,\ k = 0, \pm 1, \pm 2, \ldots)$$

then $p_1(\mu), \ldots, p_k(\mu)$ are roots of Equation (3.2) for small enough $\mu$. If $\rho_1 \neq \rho_j - ki$ $(k = 0, \pm 1, \pm 2, \ldots, j = 2, \ldots, mn)$, then $p_1(\mu) = \rho_1 + \mu^2(\ )\ldots$, that is, the characteristic exponent $p_1(\mu)$ is (to within the accuracy of small orders of $\mu$) a characteristic exponent of the system (4.1) averaged with respect to time.

2. Suppose that in the sequence (9.5), there vanish two terms and we have Equations (7.14) with $k = 0$ and $k = -\gamma$ $(\gamma > 0)$. From Lemma 7.1 and Equation (7.14) it follows that

$$\begin{aligned} L_0 (p) - L_0 (p)\, S^0_{0,0} (p) &= L_0 (p) - L_0 (p)\, S^0_{0,0-\gamma} (p) - \\ &-L_0(p)\, S^{-\gamma}_{0,0,-\gamma}(p)(L_0 (p - \gamma i) - \\ &- L_0 (p - \gamma i)\, S^{-\gamma}_{-\gamma,0,-\gamma} (p))^{-1} L_0 (p - \gamma i)\, S^0_{-\gamma,0,-\gamma} (p) \end{aligned} \tag{9.8}$$

From Note 6.3 it follows that for small enough values of $\mu$ and $\epsilon$, the series of the type (6.1) on the right side of (9.8) converge when.

$|p - p_*| < \epsilon$. In more complicated cases, when in the sequence (9.5) the zeros occur for three or more different values of $k$, one can apply Lemma 7.1. From Note 6.2 and Lemma 7.1 it follows that for the finding of the characteristic exponents of solutions one can use Lemma 9.1 also for finite matrices $A_{qj}$ $(q = \pm 1, \ldots, \pm 1, j = 0, \ldots, n)$ (4.2). Without proof we remark that Equation (9.2) constructed for Mathieu's equation [1] can be transformed with the aid of Lemma 7.1 into a continued fraction. This fraction will converge on the entire complex plane $p$, and coincides with the equation for the characteristic exponents obtained by the method of Ince in the form of a continued fraction [1] (p.32).

*Example* 2. Let us consider the system of linear differential equations with periodic coefficients

$$\frac{dY}{dt} = (A + \mu B(t))Y \tag{9.9}$$

where

$$B(t) = \sum_{k=-\infty}^{\infty} B_k e^{-ikt}, \qquad \sum_{k=-\infty}^{\infty} |B_k| < \infty \tag{9.10}$$

Let $\rho_1, \ldots, \rho_m$ be the roots of the characteristic equation (averaged with respect to time $t$) of the system (9.9) with $\mu = 0$

$$\mathrm{Det}\,(Ep - A) = 0 \tag{9.11}$$

Suppose that if $\rho_* \in \{\rho_1, \ldots, \rho_m\}$ the following conditions are satisfied

$$|p_* - \rho_s| \mp ki \qquad (k = \pm 1, \pm 2, \pm 3, \ldots, \; s = 1, \ldots, m) \tag{9.12}$$

We have

$$K_k(p) = \mu (Ep - A - \mu B_0)^{-1} B_k \tag{9.13}$$

Equation (9.2) takes on the form

$$\mathrm{Det}\Big(Ep - A - \mu B_0 - \sum_{\sigma=2}^{\infty} \mu^\sigma \sum_{\varkappa} B_{k_1} (E(p + k_1 i) - A - \mu B_0)^{-1} \ldots \tag{9.14}$$

$$\ldots B_{k_{\sigma-1}} (E(p + (k_1 + k_2 + \ldots + k_{\sigma-1}) i) - A - \mu B_0)^{-1} B_{k_\sigma}\Big) = 0$$

$$(\varkappa = k_1 + \ldots + k_\sigma = 0, 0 \in \{k_1, k_1 + k_2, \cdots, k_1, + \ldots + k_{\sigma-1}\} \qquad 0 \in \{k_1, \ldots, k^\sigma\}$$

The roots of this transcendental equation which are near $p_*$, for small values of $\mu$, will be characteristic exponents of the solutions of the system (9.9).

**10.** We shall give criteria for the stability of the solutions of a second-order equation, with real periodic coefficients, of the form

$$\frac{d^2y}{dt^2} + r(t)\frac{dy}{dt} + \left(\frac{\gamma^2}{4} + g(t)\right)y = 0 \tag{10.1}$$

where $\gamma \neq 0$ is an integer,

$$\begin{aligned} r(t) &= \sum_{k=-\infty}^{\infty} e^{ikt} r_k, \quad \sum_{k=-\infty}^{\infty} |r_k k| < c_1, \quad r_k = \bar{r}_{-k}, \quad \operatorname{Im} r_0 = 0 \\ g(t) &= \sum_{k=-\infty}^{\infty} e^{ikt} g_k, \quad \sum_{k=-\infty}^{\infty} |g_k| < c_2, \quad g_k = \bar{g}_{-k}, \quad \operatorname{Im} g_0 = 0 \end{aligned} \tag{10.2}$$

Let $p_1$ and $p_2$ be the characteristic exponents of the solution. Then it is necessary for stability that the following conditions be satisfied

$$-\operatorname{Re}(p_1 + p_2) = r_0 \geqslant 0 \tag{10.3}$$

In the considered case we have by (4.2)

$$L_0(p) = p^2 + r_0 p + \frac{\gamma^2}{4} + g_0, \qquad L_q(p) = r_{-q} p + g_{-q} \qquad (q \neq 0) \tag{10.4}$$

Let us introduce the notation

$$\begin{aligned} a_1(p) &= L_0(p) - L_0(p) S^0_{0,0,-\gamma}(p), \qquad b_1(p) = L_0(p) S^{-\gamma}_{0,0,-\gamma}(p) \\ a_2(p) &= L_0(p - \gamma i) - L_0(p - \gamma i) S^0_{0,0,\gamma}(p - \gamma i) \\ b_2(p) &= L_0(p - \gamma i) S^{\gamma}_{0,0,\gamma}(p - \gamma i) \end{aligned} \tag{10.5}$$

Multiplying the expressions (9.2) and (9.8) by $a_2(p)$ we obtain

$$a_1(p) a_2(p) - b_1(p) b_2(p) = 0 \tag{10.6}$$

Let $p = i\gamma/2 + iz$. Retaining only the linear terms of the expansion (10.5) in powers of $z$, $r_k$, $q_k$, we obtain

$$\begin{aligned} a_1\left(\frac{i\gamma}{2} + iz\right) &= \frac{i\gamma}{2} r_0 + g_0 - \gamma z + \ldots, \qquad b_1\left(\frac{i\gamma}{2} + iz\right) = \frac{i\gamma}{2} r_\gamma - g_\gamma + \ldots \\ a_2\left(\frac{i\gamma}{2} + iz\right) &= -\frac{i\gamma}{2} r_0 + g_0 + \gamma z + \ldots, \quad b_2\left(\frac{i\gamma}{2} + iz\right) = -\frac{i\gamma}{2} r_{-\gamma} - g_{-\gamma} + \ldots \end{aligned} \tag{10.7}$$

From (10.7) it follows that Equation (10.6) indeed has two roots $z_1$, $z_2$ near to zero for small values of $q_k$ and $r_k$. If $b_1(p_0) b_2(p_0) \neq 0$, where $p_0$ is a root of Equation (10.6) near $i\gamma/2$, then $a_2(p_0) \neq 0$ and the passage from (9.2) to (10.6) is valid.

Because of the property 9.1, the roots of Equation (10.6) which are near to $i\gamma/2$ will be characteristic exponents of the solutions (to within a precision of $ki$) of Equation (10.1).

Because the coefficients of Equation (10.1) are real, we have the following relations on the boundary of the region of instability where at least one of the characteristic exponents is equal to $i\gamma/2$:

$$a_1\left(\frac{\gamma i}{2}\right) = \bar{a}_2\left(\frac{\gamma i}{2}\right), \qquad b_1\left(\frac{\gamma i}{2}\right) = b_2\left(\frac{\gamma i}{2}\right) \tag{10.8}$$

Equation (10.6) takes on the form

$$a_\gamma \equiv \left|a_1\left(\frac{\gamma i}{2}\right)\right|^2 - \left|b_1\left(\frac{\gamma i}{2}\right)\right|^2 = 0 \qquad \text{for } p_0 = \frac{1}{2} i\gamma \tag{10.9}$$

If $r_0$ is increased, we find that $a_\gamma > 0$, and vice versa. From this follows the next lemma.

*Lemma 10.1.* For small enough $c_1$, $c_1$, $r_0$, and $g_0$ the equation of the boundary of the $\gamma$th region of instability, with $r_0 > 0$, will be Equation (10.9), $a_\gamma = 0$. If $r_0 > 0$, and $a_\gamma > 0$, we have stability, while for $r_0 > 0$, $a_\gamma < 0$, we have instability for the solutions of Equation (10.1). In case $r_0 = 0$, $a_\gamma = 0$, we have $p_1 = p_2 = i\gamma/2 \pmod i$, and the question on the stability cannot be answered without some auxiliary investigation.

The criterion is convenient to use in the selection of the parameters. From Lemma (10.1) it follows that the quantity $|a_1(\gamma i/2)|$ serves to stabilize, while the quantity $|b_1(\gamma i/2)|$ serves to unstabilize the system described by Equation (10.1). Equation (10.9) is applicable also for large values of $r_k$, $g_k$. In these cases it is sufficient to apply Lemma 7.1 to (10.5).

In the testing for stability it is necessary to ensure that the characteristic exponents do not take on the values $\pm(\gamma \pm 1)i/2$.

We have the following relations which are exact to within infinitesimals of the second order in $g_k$ and $r_k$:

$$a_1\left(\frac{\gamma i}{2}\right) = \frac{\gamma i}{2} r_0 + g_0 + \sum_{\substack{q=-\infty \\ q\neq 0, -\gamma}}^{\infty} \frac{1}{q(q+\gamma)}\left(r_{-q}\left(\frac{\gamma}{2}+q\right)i + g_{-q}\right)\left(\frac{\gamma i}{2} r_q + g_q\right) + \dots$$

$$b_1\left(\frac{\gamma i}{2}\right) = \frac{\gamma i}{2} r_\gamma - g_\gamma - \sum_{\substack{q=-\infty \\ q\neq 0, -\gamma}}^{\infty} \frac{1}{q(q+\gamma)}\left(r_{-q}\left(\frac{\gamma}{2}+q\right)i + g_{-q}\right)\left(-\frac{\gamma i}{2} r_{q+\gamma} + g_{q+\gamma}\right) + \dots \tag{10.10}$$

*Example 10.1.* (Taken from the work of [7]). Let us find the boundary of the region of instability of the solution (when $\mu^2 \approx 4$) of the equation

$$\frac{d^2y}{dt^2} + 2f\mu \frac{dy}{dt} + \mu^2 (1 + q \cos 2t)\, y = 0 \tag{10.11}$$

When $\mu^2 \approx 4$, we obtain

$$\gamma = 4, \quad r_0 = 2f\mu, \quad g_2 = g_{-2} = \tfrac{1}{2}\mu^2 q, \quad g_0 = \mu^2 - 4$$

$$a_1\left(\frac{\gamma i}{2}\right) = \mu^2 - 4 + 4f\mu i - \tfrac{1}{24} q^2\mu^4, \qquad b_1\left(\frac{\gamma i}{2}\right) = \tfrac{1}{16} q^2\mu^4$$

Solving Equation (10.9) for $\mu^2$, we obtain

$$\mu^2 = 4 + \tfrac{2}{3} q^2 \pm \sqrt{q^4 - 64f^2} \tag{10.12}$$

*Example 10.2.* (From the work [8]). Let us consider the equation

$$\frac{d^2x}{dt^2} + \delta \cos t \frac{dx}{dt} + \gamma_1 x = 0 \tag{10.13}$$

In this case we have

$$r_1 = r_{-1} = \frac{\delta}{2}, \qquad \gamma = 1, 2, \qquad g_0 = \gamma_1 - \frac{1}{4}, \qquad \gamma_1 - 1$$

From (10.10) and (10.9) we find the approximate equation for the first and second regions of instability.

$$\left(\gamma_1 - \tfrac{1}{4} - \tfrac{3}{32}\delta^2\right)^2 - \left(\tfrac{1}{4}\delta\right)^2 = 0 \qquad \left(\gamma - 1 - \tfrac{1}{6}\delta^2\right)^2 = 0 \tag{10.14}$$

For this equation the quantities $b_1(i\gamma/2) \equiv 0$ if $\gamma$ is an even number. In the plane of the parameters $(\gamma_1, \delta)$ there are, therefore, no even regions of instability which touch the points $\gamma_1 = n^2$, $\delta = 0$.

**11.** Let us consider the system of differential equations of the type

$$\frac{d^2Y}{dt^2} + \mu N(\theta t) \frac{dY}{dt} + (C + \mu P(\theta t))\, Y = 0 \tag{11.1}$$

where $C = (\omega_1^2, \ldots, \omega_m^2)$ is a diagonal matrix, $\omega_1^2 > \omega_2^2 > \ldots > \omega_m^2 > 0$,

$$N(t) = \sum_{k=-\infty}^{\infty} N^{(k)} e^{ikt}, \qquad N^{(k)} = \| \nu_{js}^{(k)} \|_1^m, \qquad \sum_{k=-\infty}^{\infty} | N^{(k)} k | < \infty$$

$$P(t) = \sum_{k=-\infty}^{\infty} P^{(k)} e^{ikt}, \qquad P^{(k)} = \| \pi_{js}^{(k)} \|_1^m, \qquad \sum_{k=-\infty}^{\infty} | P^{(k)} | < \infty \tag{11.2}$$

By means of the substitutions $\theta t = \tau$, $\tau = t$, $\theta = \lambda^{-1}$ we reduce the system (11.1) to the form

$$\frac{d^2Y}{dt^2} + \mu\lambda N(t)\frac{dY}{dt} + \lambda^2(C + \mu P(t))Y = 0 \tag{11.3}$$

Let $\lambda(\mu) = \lambda_0 + \mu\ (\ldots)$, $p_s(\mu)$ $(s = 1, \ldots, 2m)$ be the characteristic exponent of the system (11.3). If $\mu = 0$, then $p_s(0) = i\omega_s\lambda_0$,

$$p_{m+s}(0) = -i\omega_s\lambda_0 \qquad (s = 1, \ldots, m), \qquad \lambda(0) = \lambda_0 = \frac{1}{\theta_0} > 0$$

We shall find the $p_s(\mu)$ with a accuracy up to small orders of $\mu$.

$$\begin{aligned} L_0(p) &= Ep^2 + \mu\lambda N^{(0)} p + \lambda^2 C + \mu\lambda^2 p^{(0)} \\ L_q(p) &= \mu\lambda N^{(-q)} p + \mu\lambda^2 P^{(-q)} \qquad (q \neq 0) \end{aligned} \tag{11.4}$$

If for all $h \neq j$, or $\gamma \neq 0$ ($\gamma$ is an integer) the following conditions hold for the given $j$

$$\omega_j \pm \omega_h \neq \gamma\theta_0 \qquad (h = 1, \ldots, m) \tag{11.5}$$

then we find on the basis of Section 9 that $p_j(\mu)$ is a root (within the required order of accuracy) of the equation

$$p^2 + \mu\lambda_0\nu_{jj}^{(0)}p + \lambda^2\omega_j^2 + \mu\lambda_0^2\pi_{jj}^{(0)} = 0 \tag{11.6}$$

Suppose that the relation (11.5) is not satisfied for the given, and only for the given, $j$, $h$, $\gamma (j > h)$. Setting $p_0 = \omega_j\lambda_0$, we find that two terms, for $k = 0$ and $k = -\gamma (\gamma > 0)$, of the sequence (9.5) will vanish (become zero) if $p = p_0 i$. Let us set

$$p = ip_0 + iz \tag{11.7}$$

in Equations (9.2), (9.8).

Furthermore, instead of the words 'the element standing at the intersection of $s$th row and $r$th column' we shall simply write $\epsilon_{sr}$. Let us determine the order with respect to $\mu$ of the elements of the matrix (when $\lambda - \lambda_0 = O(\mu)$, $z = O(\mu)$)

$$L_0(p - \gamma i) - L_0(p - \gamma i)\, S^0_{0,0,-\gamma}(p - \gamma i) \tag{11.8}$$

we have

$$\varepsilon_{ss} = O(1) \qquad (s \neq h), \qquad \varepsilon_{sr} = O(\mu) \qquad (s \neq r)$$

$$\begin{aligned} \varepsilon_{hh}(z) = (\lambda^2 - \lambda_0^2)\,\omega_h^2 - 2(p_0 - \gamma)\, z &+ \\ + \mu\lambda\nu_{hh}^{(0)}(p_0 - \gamma)\, i &+ \mu\lambda^2\pi_{hh}^{(0)} = O(\mu) \end{aligned} \tag{11.9}$$

Then the order of the elements of the matrix which is the inverse of (11.8) will be

$$\varepsilon_{hh}' = \varepsilon_{hh}^{-1}(z) + O(1) = O(\mu^{-1}), \quad \varepsilon_{sr}' = O(1) \quad (s \neq h, \quad r \neq h) \tag{11.10}$$

From (6.1) and (11.4) we obtain

$$\begin{aligned} L_0(p_0 i + iz) S_{0,0,-\gamma}^{-\gamma}(p_0 i + iz) &= -\mu\lambda N^{(\gamma)}(p_0 - \gamma) i - \mu\lambda^2 P^{(\gamma)} + \mu^2(\ldots) + \ldots \\ L_0(p_0 i - \gamma i + iz) S_{-\gamma,0,-\gamma}^{0}(p_0 i + iz) & \\ &= -\mu\lambda N^{(-\gamma)} p_0 i - \mu\lambda^2 P^{(-\gamma)} + \mu^2(\ldots) + \ldots \end{aligned} \tag{11.11}$$

Let us determine the order of the elements of the matrix (9.8)

$$\varepsilon_{jj}'' = O(\mu), \qquad \varepsilon_{ss}'' = O(1) \quad (s \neq j), \qquad \varepsilon_{sr}'' = O(\mu) \quad (s \neq r)$$

If we multiply both sides of Equation (9.2) by $\epsilon_{hh}$ we obtain, just as we did in Section 10,

$$\begin{aligned} &[(\lambda^2 - \lambda_0^2)\omega_j^2 + \mu\lambda_0\nu_{jj}^{(0)} p_0 i + \mu\lambda_0^2\pi_{jj}^{(0)} - 2p_0 z]\,[(\lambda^2 - \lambda_0^2)\omega_h^2 + \\ &\quad + \mu\lambda_0\nu_{hh}^{(0)}(p_0 - \gamma) i + \mu\lambda_0^2\pi_{hh}^{(0)} - 2(p_0 - \gamma) z] \\ &= \mu^2\lambda_0^2(\nu_{jh}^{(\gamma)}(p_0 - \gamma) i + \lambda_0\pi_{jh}^{(\gamma)})(\nu_{hj}^{(-\gamma)} p_0 i + \lambda_0\pi_{hj}^{(-\gamma)}) \end{aligned} \tag{11.12}$$

Let us introduce the notation

$$\begin{aligned} a_1 &= (2\omega_j)^{-1}(2(\lambda - \lambda_0)\omega_j^2 + \mu\lambda_0\nu_{jj}^{(0)}\omega_j i + \mu\lambda_0\pi_{jj}^{(0)}) \\ a_2 &= (2(\omega_j - \gamma\theta_0))^{-1}(2(\lambda - \lambda_0)\omega_h^2 + \mu\lambda_0\nu_{hh}^{(0)}(\omega_j - \gamma\theta_0) i + \mu\lambda_0\pi_{hh}^{(0)}) \\ a_3 &= (\omega_j(\omega_j - \gamma\theta_0))^{-1}(\nu_{jh}^{(\gamma)}(\omega_j - \gamma\theta_0) i + \pi_{jh}^{(\gamma)})(\nu_{hj}^{(-\gamma)}\omega_j i + \pi_{hj}^{(-\gamma)})\mu^2\lambda_0^2 \end{aligned} \tag{11.13}$$

Taking into account (11.7), we obtain from (11.12) the values of the characteristic exponents which have merged at the point $p = ip_0$ when $\mu = 0$:

$$p_{1,2}(\mu) = ip_0 + \tfrac{1}{2} i\left(a_1 + a_2 \pm \sqrt{(a_1 - a_2)^2 + a_3}\right) \qquad (\mathrm{mod}\ i) \tag{11.14}$$

A necessary condition for the stability of the solutions of the system (11.1) is

$$\mathrm{Im}\,(a_1 + a_2) \geqslant \mathrm{Im}\,\sqrt{(a_1 - a_2)^2 + a_3} \tag{11.15}$$

The case when the condition (11.5) is violated for several values $h$ and $\gamma$, has to be treated separately.

*Example 11.1.* Let us consider the canonical case [9, 10] of the system (11.1) when $N(t) \equiv 0$, $P^*(t) = P(t)$; here the asterisk denotes the adjoint matrix

$$\begin{aligned} a_1 &= (2\omega_j)^{-1}\,(2\,(\lambda-\lambda_0)\,\omega_j^{\,2}+\mu\lambda_0\pi_{jj}^{\,(0)}) \\ a_2 &= (2\,(\omega_j-\gamma\theta_0))^{-1}\,(2(\lambda-\lambda_0)\,\omega_h^{\,2}+\mu\lambda_0\pi_{hh}^{\,(0)}) \\ a_3 &= (\omega_j\,(\omega_j-\gamma\theta_0))^{-1}\,\mu^2\lambda_0^2\pi_{jh}^{\,(\gamma)}\pi_{hj}^{\,(-\gamma)} \qquad (\pi_{jh}^{\,(\gamma)}=\bar{\pi}_{hj}^{\,(-\gamma)}) \end{aligned} \tag{11.16}$$

From (11.14) it follows that if $\omega_j - \gamma\theta_0 > 0$, that is, if $\theta = (\omega_j - \omega_h)/\gamma$, then the characteristic exponents are pure imaginary and distinct; namely, we have the case of stability. This follows from the theorem of Krein [9, 10]. If $\omega_j - \gamma\theta_0 < 0$, that is, if $\theta_0 = (\omega_j + \omega_h)/\gamma$, then on the boundary of the region of instability the characteristic exponents must coincide; namely,

$$(a_1 - a_2)^2 + a_3 = 0 \tag{11.17}$$

By means of the substitutions $\lambda = \theta^{-1}$, $\lambda_0 = \theta_0^{-1}$, and $\omega_j - \gamma\theta_0 = \omega_h$, we obtain the equation of the boundary of the region of instability [10] (the formula of Malkin)

$$\theta = \theta_0 + \frac{\mu}{2\gamma}\left[\frac{\pi_{jj}^{\,(0)}}{\omega_j} + \frac{\pi_{hh}^{\,(0)}}{\omega_h} \pm \frac{2\,|\,\pi_{jh}^{\,(\gamma)}\,|}{\sqrt{\omega_j\omega_h}}\right] \tag{11.18}$$

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